

TOPOLOGICAL CONJUGATIONS ARE NOT CONSTRUCTABLE

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ABSTRACT. We construct two computable topologically conjugate functions for which no conjugacy is computable, or even hyperarithmetic, resolving an open question of Kennedy and Stockman [Pea07].

1. INTRODUCTION

Let I denote the closed unit interval. Two continuous functions $f, g : I \rightarrow I$ are *topologically conjugate* if there is a homeomorphism h of I such that $f \circ h = h \circ g$. The function h is called a *topological conjugation* or a *conjugacy*. In [Pea07, pg. 298] Ingram records the following question of Kennedy and Stockman: Given f and g which are topologically conjugate, how can one construct a conjugacy? If constructing a conjugacy h means providing an algorithm which computes arbitrarily good approximations to h , we show that in general there is no such construction.

Proposition 3.4. *There are two computable topologically conjugate functions with no computable conjugacy.*

It is possible to force conjugacies to be much less constructable than merely noncomputable. The hyperarithmetic functions, a superset of the computable functions, include any function that can be “constructed” as the result of a transfinite computation of countable ordinal length, where the order type of the computation length must be computable as a linear order in the sense of Section 2.1. For an introduction to the hyperarithmetic hierarchy we refer the reader to [Sac90].

Proposition 4.5. *There are two computable topologically conjugate functions with no hyperarithmetic conjugacy.*

Both constructions rely centrally on the fact that any topological conjugacy of f and g must include an order isomorphism from the fixed points of f to the fixed points of g . We use pairs of computable linear orderings without any computable order isomorphism to specify the fixed points of f and g .

In Section 2 we define the needed notions from computability theory. In Section 3 we construct two computable topologically conjugate functions with no computable conjugacy. In Section 4, we build on the methods of Section 3 to construct two computable topologically conjugate functions with no hyperarithmetic conjugacy.

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2. PRELIMINARIES

We cover the notation and basic computability concepts for infinite binary sequences in Section 2.1, trees in Section 2.2, and real-valued functions in Section

2.3. This section contains all the background needed for Sections 3 and for the construction in Section 4.2.

2.1. Computability in Cantor space. *Cantor space*, denoted 2^ω , is the set of all infinite sequences of 0's and 1's. If $X \in 2^\omega$, then $X(n)$ refers to the n th element of X . For $X, Y \in 2^\omega$, we say that $X < Y$ if $X \neq Y$ and $X(n) < Y(n)$ where n is the first place where they differ. Elements of Cantor space are also identified in the natural way with subsets of the natural numbers.

A set $X \in 2^\omega$ is *computable* if there is an algorithm which, on input n , outputs $X(n)$. Formally, algorithms are represented as *Turing machines*. A Turing machine accepts natural numbers as inputs. On a given input, a Turing machine may output a natural number, or it may run forever. If Γ is a Turing machine, then $\Gamma(n)$ denotes its output on input n , if this output exists. For more details about Turing machines we refer the reader to [Soa87].

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is called *computable* if there is a Turing machine Γ such that $\Gamma(n) = f(n)$ for all n .

A set X is called *computably enumerable* if X is empty or if there is a Turing machine Γ which halts on all its inputs such that $X = \{\Gamma(n) : n \in \mathbb{N}\}$. In this case the sequence $\Gamma(0), \Gamma(1), \Gamma(2), \dots$ is called an *enumeration* of X . We will use the following fact:

Fact 2.1. *There is a computably enumerable set A which is not computable.*

An *oracle Turing machine* is a Turing machine which is permitted to access arbitrary bits of a set called the *oracle* as a part of its computation. The oracle is an element of 2^ω . The output of an Turing machine Γ with oracle X on input n is denoted $\Gamma^X(n)$.

If $X, Y \in 2^\omega$, we say X *computes* Y if there is an oracle Turing machine Γ such that $\Gamma^X(n) = Y(n)$ for all n . If X computes Y and X is computable, then Y is also computable.

A function $f : 2^\omega \rightarrow 2^\omega$ is called *computable* if there is an oracle Turing machine Γ such that for all X and n , $\Gamma^X(n) = f(X)(n)$. In this case, we say that $f(X)$ is *uniformly computable* from X .

There is a bijective *pairing function* $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, whose inverse is computable in the sense that the maps $\langle n, m \rangle \mapsto n$ and $\langle n, m \rangle \mapsto m$ are computable. This function is useful for encoding information into subsets of \mathbb{N} . It is also useful for combining the information from multiple $X \in 2^\omega$ in an orderly way. Given $\{X_n\}_{n \in \mathbb{N}}$ such that each $X_n \in 2^\omega$, we write $\bigoplus_n X_n$ to denote the set $Y \in 2^\omega$ such that $Y(\langle n, m \rangle) = 1$ if and only if $m \in X_n$.

Functions on natural numbers can be encoded into subsets of \mathbb{N} as follows. The code for a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is the set $\{\langle n, f(n) \rangle : n \in \mathbb{N}\}$. This f is computable (computability for functions from \mathbb{N} to \mathbb{N} is defined above) if and only if its code is a computable subset of \mathbb{N} .

Relations on natural numbers can also be encoded into subsets of \mathbb{N} . In Section 4 we consider linear orders on subsets of the natural numbers. A linear order $(A, <_R)$ can be encoded as $\{\langle a, b \rangle : a, b \in A \text{ and } a <_R b\}$. A linear order is *computable* if and only if its code is computable.

For more information about computability theory, we refer the reader to [Soa87].

2.2. Trees and Finite Binary Strings. We write $2^{<\omega}$ for the set of all finite binary strings. A tree is a subset of $2^{<\omega}$ which is closed under taking initial segments. The empty string is denoted \emptyset . If $\sigma \in 2^{<\omega}$, we write $|\sigma|$ for the length of σ , and $\sigma(n)$ for the n th element of σ , where n starts at 0. If $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \subseteq \tau$ if $|\sigma| \leq |\tau|$ and for each $n < |\sigma|$, $\sigma(n) = \tau(n)$. We write $\sigma \hat{\ } \tau$ to denote the concatenation of σ and τ . By $\sigma \hat{\ } \bar{1}$ (respectively $\sigma \hat{\ } \bar{0}$) we mean the real $X \in 2^\omega$ such that $X(n) = \sigma(n)$ if $n < |\sigma|$ and $X(n) = 1$ (respectively $X(n) = 0$) otherwise.

Trees can be encoded into subsets of \mathbb{N} as follows. There is a bijective correspondence between $2^{<\omega}$ and \mathbb{N} which causes the basic functions and relations on finite strings defined above to be computable. The code for a tree is the subset of \mathbb{N} consisting of the codes for each of its finite strings. A tree is called *computable* if its code is computable.

If $X \in 2^\omega$, $X \upharpoonright n$ denotes the finite binary string which is the first n bits of X . If T is a tree, then $[T] \subseteq 2^\omega$ is the set of all X such that $X \upharpoonright n \in T$ for all n . The set $[T]$ is also called the path set of T and its elements are called paths.

2.3. Computability for Reals and Real-Valued Functions. A real number in $[0, 1]$ is computable if and only if its binary decimal expansion is computable. However, we do not use binary expansions to represent real numbers because these expansions behave badly near dyadic rationals. (Knowing that $x \in (.5 - \varepsilon, .5 + \varepsilon)$ gives us no information about the first digit of x 's binary decimal expansion, no matter how small ε is.)

The rational numbers can be encoded into the natural numbers as follows. A code for a rational number q is a natural number $m = \langle s, \langle a, b \rangle \rangle$ such that $q = (-1)^s \frac{a}{b}$.

A code for a real number r is meant to encode a Cauchy sequence of rationals $\{q_n\}_{n \in \mathbb{N}}$ converging to r . Formally, a *code* for r is a set $X \subseteq 2^\omega$ such that

- (1) for each $n \in \mathbb{N}$, there is exactly one m such that $\langle n, m \rangle \in X$
- (2) this m is a code for a rational number q such that $|q - r| < 2^{-n}$.

Note that there are many different codes for each real. A real is *computable* if it has a computable code. This means that a real r is computable if and only if there is an algorithm which, on input n , returns a rational approximation to r that is accurate to within 2^{-n} .

A code for a continuous function $f : I \rightarrow I$ is meant to specify f on a dense subset of I and provide a modulus of uniform continuity. Let q_1, q_2, \dots be a computable enumeration of codes for all the rational numbers in I . A code for a continuous function $f : I \rightarrow I$ is a set $X = \bigoplus_n X_n$ such that

- (1) for each $i > 0$, X_i is a code for $f(q_i)$
- (2) X_0 is a code for a function $d : \mathbb{N} \rightarrow \mathbb{N}$ such that $|f(x) - f(y)| < 2^{-m}$ whenever $|x - y| < 2^{-d(m)}$.

A continuous function $f : I \rightarrow I$ is *computable* if and only if it has a computable code. Finite sums and products of computable functions are computable.

For more information about computable analysis we refer the reader to [PER89].

3. NONCOMPUTABLE CONJUGATION

In this section we construct two computable functions $f, g : [0, 1] \rightarrow [0, 1]$ such that f and g are topologically conjugate but not computably topologically conjugate. Both f and g are homeomorphisms of $[0, 1]$ satisfying $f(x) \geq x$ and $g(x) \geq x$ for all x . Each of them has a set of fixed points corresponding to the set of paths

through a computable tree. The trees' path sets are order isomorphic, but have no computable order isomorphism. Since any conjugacy induces an order isomorphism, no conjugacy can be computable.

First we specify the locations where it is possible for our functions to have a fixed point. The following Cantor set differs from the canonical one in that it has been shrunk to fit in the interval $[\frac{1}{3}, \frac{2}{3}]$.

Definition 3.1. Let $\mathcal{C} \subset [0, 1]$ denote the set of x whose ternary decimal expansion begins with 1 and continues using only 0 and 2. If $X \in 2^\omega$, let $c(X)$ denote the unique $x \in \mathcal{C}$ whose first ternary digit is 1 and whose $n+1$ st ternary digit is $2X(n)$.

Observe that c is an order isomorphism between \mathcal{C} and 2^ω . Also, considering c as a function from 2^ω to $\{X \in 2^\omega : X \text{ is a code for some } x \in \mathcal{C}\}$, both c and c^{-1} are computable.

Lemma 3.1. From any (code for a) computable tree $T \subseteq 2^{<\omega}$, one may uniformly compute a (code for a) computable $f_T : [0, 1] \rightarrow [0, 1]$ satisfying

- (1) f_T continuous and increasing with $f_T(x) \geq x$ for all x
- (2) $f_T(x) = x$ if and only if $x \in \{0, 1\}$ or $x \in \mathcal{C}$ and $c^{-1}(x) \in [T]$.

Proof. Let $b : [0, 1] \rightarrow [0, 1]$ be a smooth bump function which satisfies $b(0) = b(1) = b'(0) = b'(1) = 0$ and for all x , $b'(x) < 1$. For example, we could have $b(x) = \frac{1}{K} e^{-\frac{1}{x(1-x)}}$ where K is a constant chosen large enough to guarantee that for all x , $b'(x) < 1$.

Given T , we build a sequence of functions h_n as follows. Let C_n denote the set of all $\sigma \in T$ such that $|\sigma| = n$. To each such σ , associate the closed interval $I_\sigma = [c(\sigma^\frown 0), c(\sigma^\frown 1)]$. Let $U_n = (0, 1) \setminus \bigcup_{\sigma \in C_n} I_\sigma$. Then U_n is an open set consisting of finitely many connected components $(a_1, c_1), \dots, (a_r, c_r)$. Let

$$h_n(x) = \sum_{i=1}^r \frac{1}{3^n} b[a_i, c_i](x),$$

where

$$b[a, c](x) = \begin{cases} (c-a)b\left(\frac{x-a}{c-a}\right) & \text{if } x \in [a, c] \\ 0 & \text{otherwise,} \end{cases}$$

that is, $b[a, c]$ is the function b scaled proportionally so that it is supported on $[a, c]$. Then define

$$f_T = x + \sum_{n=1}^{\infty} h_n(x).$$

The function f_T is continuous because it is the uniform limit of continuous functions. Note also that each h_n is differentiable with $|h'_n(x)| < 3^{-n}$ for each n (the bumps that comprise h_n have disjoint support). Therefore, the sum $1 + \sum_{n=1}^{\infty} h'_n(x)$ also converges uniformly to a function g satisfying $\frac{1}{2} < g(x) < \frac{3}{2}$ for all x . So f_T is differentiable with $f'_T(x) = g(x) > 0$ for all x , so f_T is increasing. And because $h_n(x) \geq 0$ for all x , $f_T(x) \geq x$ for all x . It is clear that $f_T(0) = 0$ and $f_T(1) = 1$. And for $x \in (0, 1)$, $x \in \bigcup_n U_n$ if and only if $x \notin c([T])$, so $f(x) = x$ for $x \in c([T])$ and $f(x) > x$ otherwise.

Finally, f_T is computable because the following approximation holds:

$$\left| f_T(x) - \left(x + \sum_{n=1}^N h_n(x) \right) \right| < \sum_{n=N+1}^{\infty} 3^{-n} = \frac{1}{2 \cdot 3^N}.$$

This can be used to compute $f_T(q)$ to any precision for any rational q . Furthermore, $|f'_T(x)| < \frac{3}{2}$ for all x , so $|f_T(x) - f_T(y)| < \frac{3}{2}|x - y|$, which gives a computable modulus of continuity. \square

Lemma 3.2. *Let $P, Q \subseteq 2^{<\omega}$. The following are equivalent:*

- (1) f_P and f_Q are topologically conjugate.
- (2) $\text{ot}([P]) = \text{ot}([Q])$

Furthermore, if h is a homeomorphism of $[0, 1]$ such that $h \circ f_P = f_Q \circ h$, then h computes an order isomorphism $h^* : [P] \rightarrow [Q]$.

Proof. First note that if $h \circ f_P = f_Q \circ h$, then h is necessarily order preserving. If h were order reversing, consider any x such that $f_P(x) > x$. Then $f_P(x) = h^{-1}(f_Q(h(x))) > x$, so $f_Q(h(x)) < h(x)$, which is impossible.

Suppose $h \circ f_P = f_Q \circ h$ as above. For $X \in [P]$, define $h^*(X) = c^{-1}(h(c(X)))$. Since $c(X)$ is a fixed point of f_P , $h(c(X))$ is a fixed point of f_Q . Since $c(X) \notin \{0, 1\}$, $h(c(X)) \notin \{0, 1\}$, so $h(c(X)) = c(Y)$ for some $Y \in [Q]$. Therefore h^* is well-defined. By a similar argument, h^* is onto. Because c and h are order preserving, so is h^* . Because c and c^{-1} are computable, h^* is computable from h .

On the other hand, suppose $h^* : [P] \rightarrow [Q]$ is an order isomorphism. Define h as follows. For $x \in c([P])$, the fixed points of f_P , define $h(x) = c(h^*(c^{-1}(x)))$, the corresponding fixed point of f_Q . Set $h(0) = 0, h(1) = 1$.

Now if x is not a fixed point of f_P , x lies in an interval (a, b) such that $f_P(a) = a, f_P(b) = b$, and $f_P(z) > z$ for $z \in (a, b)$. Given such an interval (a, b) , define $h \upharpoonright (a, b)$ as follows. Fix some $x_0 \in (a, b)$, and some $y_0 \in (h(a), h(b))$. For all $z \in (a, b)$, $f_P(z) \in (a, b)$, since f_P is order preserving and $f_P(b) = b$. Furthermore, $\lim_{n \rightarrow \infty} f_P^n(z) = b$, because this limit must be a fixed point of f_P (apply f_P to both sides). Now since f_P is a strictly increasing function, its inverse is well-defined, and by similar arguments to the previous, $z > f_P^{-1}(z) > \dots > f_P^{-n}(z) > \dots$ with $\lim_{n \rightarrow \infty} f_P^{-n}(z) = a$. The analogous facts hold for the interval $(h(a), h(b))$ with respect to the function f_Q . Let $i : [x_0, f_P(x_0)) \rightarrow [y_0, f_Q(y_0))$ be any homeomorphism. For $x \in (a, b)$, define

$$h(x) = f_Q^n(i(f_P^{-n}(x)))$$

where $n \in \mathbb{Z}$ is the unique integer such that $x \in [f_P^n(x_0), f_P^{n+1}(x_0))$.

Observe that h maps $[f_P^n(x_0), f_P^{n+1}(x_0))$ homeomorphically onto $[f_Q^n(y_0), f_Q^{n+1}(y_0))$. By the arrangement of these intervals, h is continuous at their endpoints, so h maps (a, b) homeomorphically onto $(h(a), h(b))$. In the same way we see that h is continuous at the fixed points of f_P , and so $h : [0, 1] \rightarrow [0, 1]$ is a homeomorphism.

We claim that $h \circ f_P = f_Q \circ h$. For the fixed points of f_P this is clear. Given x not a fixed point, let $(a, b) \ni x$ and x_0 be as in the definition of h . Then

$$h(f_P(x)) = f_Q^n(i(f_P^{-n}(f_P(x)))) = f_Q(f_Q^{n-1}(i(f_P^{-(n-1)}(x)))) = f_Q(h(x))$$

where $n \in \mathbb{Z}$ is the unique integer such that $f_P(x) \in [f_P^n(x_0), f_P^{n+1}(x_0))$. \square

Lemma 3.3. *There are two computable trees P and Q such that $\text{ot}([P]) = \text{ot}([Q])$ but there is no computable $h^* : [P] \rightarrow [Q]$ witnessing the isomorphism.*

Proof. Let $P = \{\underbrace{1 \dots 1}_n \underbrace{0 \dots 0}_m : n, m \in \omega\}$. Then $\text{ot}([P]) = \omega + 1$. Let A be any computably enumerable set which is not computable. Since A is not computable,

the complement of A is infinite. Let $a_0, a_1, \dots, a_n, \dots$ be an enumeration of the elements of A . Let $Q = \{\underbrace{1 \dots 1}_n \underbrace{0 \dots 0}_m : n, m \in \omega, n \neq a_i \text{ for any } i < m\}$. Since A is coinfinite, $\text{ot}([Q]) = \omega + 1$ as well. Suppose $h^* : [P] \rightarrow [Q]$ is an order isomorphism. Then $h^*(\bar{1}) = \bar{1}$ and $h^*(\underbrace{1 \dots 1}_n \underbrace{1 \wedge \bar{0}}_m) = \underbrace{1 \dots 1}_m \bar{0}$ where m is the n th element in the complement of A . Thus h^* computes A (to find the n th element of $\mathbb{N} \setminus A$, compute the bits of $h^*(\underbrace{1 \dots 1}_n \underbrace{1 \wedge \bar{0}}_m)$ until the first 0 appears). So h^* is not computable. \square

Proposition 3.4. *There exist two computable topologically conjugate functions with no computable conjugacy.*

Proof. Let P and Q be as in Lemma 3.3. By Lemma 3.2, f_P and f_Q are topologically conjugate, but any conjugation computes an order isomorphism between $[P]$ and $[Q]$, and is thus not computable. \square

4. NON-HYPERARITHMETIC CONJUGATION

We strengthen the result of the previous section by constructing two topologically conjugate functions with no hyperarithmetic conjugacy.

No background beyond the material in Section 2 is needed to understand most of the material in Section 4.2, which contains the construction. The exception is Lemma 4.4, which assumes Corollary 4.2, familiarity with the Turing jump, and the fact that if $X \in 2^\omega$ is hyperarithmetic, then anything computable from the jump of X is also hyperarithmetic. Sections 4.1 and 4.3 assume familiarity with the hyperarithmetic hierarchy. For an introduction to hyperarithmetic theory, we refer the reader to [Sac90].

4.1. Isomorphic Computable Linear Orders With No Hyperarithmetic Isomorphism. By \mathcal{O}^* we mean the set of notations for recursive linear orderings with no hyperarithmetic descending sequences, which is defined in [FS62] and further developed in [Har68]. It is defined as follows:

$$\mathcal{O}^* = \cap \{X : X \in \text{HYP} \wedge 1 \in X \wedge (z \in X \rightarrow 2^z \in X) \wedge (\forall n[\phi_e(n) \in X \wedge \phi_e(n) <_{\mathcal{O}^*} \phi_e(n+1)] \rightarrow 3 \cdot 5^e \in X)\},$$

where ϕ_e is the e th Turing functional and $<_{\mathcal{O}^*}$ is the computably enumerable relation satisfying $1 <_{\mathcal{O}^*} x$ if $x \neq 1$; $z <_{\mathcal{O}^*} 2^z$; $\phi_e(n) <_{\mathcal{O}^*} 3 \cdot 5^e$; and $a <_{\mathcal{O}^*} b \wedge b <_{\mathcal{O}^*} c \rightarrow a <_{\mathcal{O}^*} c$.

Harrison [Har68] proved the following structure theorem for $\mathcal{O}^* \setminus \mathcal{O}$, the non-standard ordinal notations:

Theorem 4.1 (Harrison). *Suppose $a \in \mathcal{O}^* \setminus \mathcal{O}$. Let $1 + \eta$ be the order type of the rationals in $[0, 1)$. Then there exists a unique $\alpha < \omega_1^{CK}$ such that $\{y : y <_{\mathcal{O}^*} a\}$ has order type $\omega_1^{CK} \cdot (1 + \eta) + \alpha$.*

Our use of this structure theorem is limited to the following corollary:

Corollary 4.2. *There exists a pair of isomorphic computable linear orderings such that no isomorphism between them is hyperarithmetic.*

Proof. Let $a \in \mathcal{O}^* \setminus \mathcal{O}$ with $\text{ot}(\{y : y <_{\mathcal{O}^*} a\}) = \omega_1^{CK} \cdot (1 + \eta)$. Then there is another nonstandard notation $b <_{\mathcal{O}^*} a$ whose standard part is also 0. So $\text{ot}(\{y : y <_{\mathcal{O}^*} a\}) = \text{ot}(\{y : y <_{\mathcal{O}^*} b\})$. For contradiction, suppose that $i : \{y : y <_{\mathcal{O}^*} a\} \rightarrow \{y : y <_{\mathcal{O}^*} b\}$ is a hyperarithmetic isomorphism. Then $\{i^n(a)\}_n$ is a hyperarithmetic descending sequence in $\{y : y <_{\mathcal{O}^*} a\}$, contradicting that $a \in \mathcal{O}^*$. \square

4.2. Construction of Topologically Conjugate Functions. We encode arbitrary computable linear orders into the path sets of computable trees, guaranteeing the path sets are order isomorphic if and only if the orders were.

Lemma 4.3. *Uniformly in any linear ordering $R = (A, <_R)$, one may compute a tree $T_R \subseteq 2^{<\omega}$ and a labeling function $l_R : A \rightarrow T_R$ such that*

- (1) *For each $\sigma \in T$, the last bit of σ is 0 if and only if $\sigma = l_R(a)$ for some $a \in A$.*
- (2) *For each $a \in A$, $l_R(a) \frown \bar{1} \in [T_R]$*
- (3) *For each $a, b \in A$, $a <_R b$ if and only if $l_R(a) \frown \bar{1} < l_R(b) \frown \bar{1}$.*
- (4) *If $X \in [T_R]$, then $X = l_R(a) \frown \bar{1}$ for some $a \in A$ if and only if X has a successor in $[T_R]$.*
- (5) *The order type of $[T_R]$ depends only on the order type of R .*
- (6) *Two linear orders R and S are isomorphic if and only if the associated $[T_R]$ and $[T_S]$ are isomorphic.*

Proof. Construct T_R and l_R in stages as follows. At stage n we decide which strings σ of length n belong to T_R , and define $l_R(a)$ for all elements a which have been seen to be in A by stage n .

At stage 0 put the empty string in T_R . At stage $n + 1$, for each $\tau \in T_R$ such that $|\tau| = n$, put $\tau \frown 1$ into T_R . If no new element of A has been enumerated in this time, the stage is completed. Otherwise, there are two possibilities. If the new element a is R -bigger than any element that has been enumerated previously, put $\sigma = \underbrace{1 \dots 1}_n \frown 0$ in T_R and let $l_R(a) = \sigma$. On the other hand, if b is least among

the previously enumerated elements of A such that $b >_R a$, then already we have $\tau = l_R(b) \frown \underbrace{1 \dots 1}_{n - |l_R(b)|} \in T_R$. Put $\tau \frown 0$ in T_R and define $l_R(a) = \tau \frown 0$. This completes

stage $n + 1$. The tree T_R and the function l_R have now been defined.

The first three parts of the lemma follow directly from the construction. For part 4, suppose that $X = l_R(a) \frown \bar{1}$. Then X has a successor in $[T_R]$: the leftmost path of T_R that begins with $(l_R(a) \upharpoonright (|l_R(a)| - 1)) \frown 1$. On the other hand, suppose that X does not take this form. Then either $X = \bar{1}$, in which case it does not have a successor, or X has infinitely many zeros, in which case it has infinitely many labeled substrings $l_R(a_1) \subset l_R(a_2) \subset \dots \subset X$. Then we have $l_R(a_1) \frown \bar{1} > l_R(a_2) \frown \bar{1} > \dots$ with $\lim_{n \rightarrow \infty} l_R(a_n) \frown \bar{1} = X$, so X has no successor in $[T_R]$.

The order type of $[T_R]$ can be described as follows. Let \mathcal{U} be the set of all upward closed subsets of A that have no least element. We define an ordering $<^*$ on $A \cup \mathcal{U}$ which extends $<_R$. If $U \in \mathcal{U}$ and $a \in A$, say that $a <^* U$ if and only if $a \notin U$. If $U, V \in \mathcal{U}$, say that $U <^* V$ if and only if $U \setminus V$ is nonempty. Then consider the order $\langle A \cup \mathcal{U}, <^* \rangle$. This new order has an order type which depends only on the order type of $\langle A, <_R \rangle$. Furthermore, this is the same order type that the tree $[T_R]$ has, via the order preserving bijection $a \mapsto l_R(a) \frown \bar{1}$, $U \mapsto \inf(\{\bar{1}\} \cup \{l_R(a) \frown \bar{1} : a \in U\})$.

Finally, an isomorphic copy of R may be recovered from $[T_R]$ by restricting the domain of the latter to $\{X : X \text{ has a successor in } [T_R]\}$. Thus if S and R are linear orders, $[T_R]$ and $[T_S]$ are isomorphic if and only if R and S are. \square

Next we will see that from a isomorphism between path sets of such trees, an isomorphism between the original orders may be obtained (using one Turing jump). Thus, the pair of isomorphic linear orders from Corollary 4.2 generates a pair of trees whose isomorphic path sets have no hyperarithmetical isomorphism.

Lemma 4.4. *There exist computable trees $P, Q \subseteq 2^{<\omega}$ such that $\text{ot}([P]) = \text{ot}([Q])$ but there is no hyperarithmetical $h : [P] \rightarrow [Q]$ which witnesses the isomorphism.*

Proof. Let $R = (A_R, <_R)$ and $S = (A_S, <_S)$ be two isomorphic computable linear orderings such that no isomorphism between them is hyperarithmetical. Compute l_R, l_S, T_R and T_S as in Lemma 4.3. Let $h : [T_R] \rightarrow [T_S]$ be an order isomorphism. We claim that there is an order isomorphism $h^* : A_R \rightarrow A_S$ which is computable in the jump of h . By considering only those elements of $[T_R]$ and $[T_S]$ which have successors, we see that the restriction $h : \{l_R(a) \smallfrown \bar{1} : a \in A_R\} \rightarrow \{l_S(b) \smallfrown \bar{1} : b \in A_S\}$ is an order isomorphism. Given $a \in A_R$, compute $h^*(a)$ as follows. Consider the enumeration, computable in h , of the bits of $h(l_R(a) \smallfrown \bar{1})$. Each time a zero appears in that enumeration, ask the h -jump oracle if there will be another zero. Eventually the last zero will be found. At that point σ has been enumerated with $\sigma(|\sigma|) = 0$ and $\sigma = l_S(b)$ for some $b \in A_S$. Return this b . (Since l_S is recursive, the search for b such that $l_S(b) = \sigma$ is guaranteed to terminate.) Because $a \mapsto l_R(a) \smallfrown \bar{1}$, $h \upharpoonright \{l_R(a) \smallfrown \bar{1} : a \in A_R\}$, and $b \mapsto l_S(b) \smallfrown \bar{1}$ are each order isomorphisms, h^* is an order isomorphism. Since h^* is not hyperarithmetical, h is not hyperarithmetical either. \square

Proposition 4.5. *There exist two computable topologically conjugate functions with no hyperarithmetical conjugacy.*

Proof. Let P and Q be as in Lemma 4.4. By Lemma 3.2, f_P and f_Q are topologically conjugate, but any conjugation computes an order isomorphism between $[P]$ and $[Q]$, and is thus not hyperarithmetical. \square

4.3. Discussion. The above construction reduces pairs of linear orders to pairs of $C[0, 1]$ functions such that the functions are topologically conjugate if and only if the orders were isomorphic.

Corollary 4.6. *The set of all (pairs of indices for) computable topologically conjugate pairs of functions is Σ_1^1 -complete.*

Proof. Due to the continuity of all functions involved, f and g are topologically conjugate if and only if

$$(1) \quad \exists h [h \text{ is a homeomorphism of } I \text{ and } (\forall x \in \mathbb{Q} \cap I) [f(h(x)) = h(g(x))]].$$

Note that the matrix is arithmetic. Therefore, the statement “ f and g are topologically conjugate” is Σ_1^1 .

On the other hand, it is known that the isomorphism problem for computable linear orders is Σ_1^1 -complete. Proofs are given in, for example, [GK02, Theorem 4.4(d)] and [CDH08, Lemma 5.2]. We have demonstrated a computable reduction from the isomorphism problem for linear orders to the conjugacy problem for functions on the interval (Lemmas 4.3 and 3.1). Therefore the conjugacy problem is Σ_1^1 -complete. \square

Furthermore, by (1) the set of all h such that h is a conjugation of f and g is Δ_1^1 relative to f and g . Therefore, assuming f and g are computable (or even hyperarithmetical) and topologically conjugate, they must have an \mathcal{O} -computable conjugacy. In fact, the Gandy basis theorem guarantees the existence of a hyperarithmetically low conjugacy. Therefore, the result in Proposition 4.5 is as strong as possible.

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